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Radiation Patterns and Radiation Fields

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RADIATION PATTERNS AND RADIATION FIELDS

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Abstract

This paper deals with the asymptotic behavior at infinity of the solutions of the scalar wave equation which satisfy a radiation condition. A radiation pattern is defined to be the angular factor of the leading term in the asymptotic expansion. Not every function can be such a pattern. Necessary and sufficient conditions for a given function to be a radiation pattern are derived. These conditions also enable one to determine the smallest possible sphere that can contain all the sources for a given pattern.

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1. Introduction

Consider a scalar radiation field, described by a function U, which satisfies

$$\Delta U + k^2 U = 0$$

with positive real k outside a certain finite region G. Let this function U satisfy the Sommerfeld radiation conditions. This function then may be regarded as being generated by volume sources, surface sources, or point singularities all of which are inside G.

The radiation pattern of the field is then given by the asymptotic behavior of the function U at infinity. The object of this paper is to investigate the relation between the radiation pattern and the function U. We shall prove first that when we introduce spherical coordinates $(x) = R(\xi)$, any function which has the properties mentioned above satisfies, for $R \to \infty$, an asymptotic relation

$$U(r\xi) \simeq \frac{e^{ikR}}{R} f(\xi) ,$$

where $f(\xi)$ can be determined. This function $f(\xi)$, which depends on the direction vector (ξ) only, is defined to be the radiation pattern, and we may call it the pattern of U.

We consider next the inverse problem: to determine whether for a given $f(\xi)$ there exists a function U such that $f(\xi)$ is the radiation pattern of U. In other words, we wish to determine the possible patterns. We are able to give necessary and sufficient conditions for a function of $f(\xi)$ to be a pattern of some U. We can do even more and give an estimate for the location of the sources; then we know that certain patterns, whose sources or singularities are too close together, cannot be generated.

This, of course, is primarily a mathematical problem because we require that the asymptotic relation hold exactly for all directions; it will be shown that if we were content with getting this expansion with a function $f_1(\xi)$ which approximates $f(\xi)$, the problem would be much easier.

The main result of this paper is a criterion for determining whether or not a prescribed function $f(\xi)$ is admissible as a pattern. This criterion also enables us to determine the radius of the smallest sphere containing the sources which generate the pattern.

2. Formulation of the Problem

Let the vector of coordinates in a three-dimensional Euclidean space be denoted by (x) and let

$$(x) = R(\xi) \ , \tag{1}$$

where (ξ) is the unit vector in the direction of (x). Let $U(x)$ be a solution of

$$\Delta U + k^2 U = 0 \ , \tag{2}$$

where k is a positive constant such that for $|x| \geq C$, $U(x)$ satisfies the Sommerfeld radiation condition

$$\lim_{R \to \infty} R\left(\frac{\partial U}{\partial n} - ikU\right) = 0 \tag{3}$$

uniformly for all directions.

The purpose of this paper, then, is to discuss the asymptotic relation

$$U(R\xi) \simeq \frac{e^{ikR}}{R} f(\xi) \tag{4}$$

or more precisely

$$U(R\xi) = \frac{e^{ikR}}{R} f(\xi) + O\left(\frac{1}{R^2}\right) \ , \tag{5}$$

where the function $f(\xi)$ depends on the direction only. This function $f(\xi)$ is defined to be the radiation pattern of the function $U(x)$.

There are two aspects of this problem of radiation patterns. The first is to prove (5) and express $f(\xi)$ in terms of $U(x)$; the second is to find a function $U(x)$ having a given radiation pattern $f(\xi)$. This last problem involves determining the class of functions $f(\xi)$, defined on the unit sphere, which may occur as radiation patterns. This class is not the class of continuous functions nor is it the class of analytic functions; it can best be characterized by considering an associated class of harmonic functions $H(x)$. The order of magnitude of $H(x)$ gives a criterion as to whether or not $f(\xi)$ can be a radiation pattern.

We formulate and prove these properties in

Theorem 1. A necessary and sufficient condition for a function $f(\xi)$ defined on a unit sphere to be a radiation pattern is that there exist a harmonic function $H(x)$ which is analytic for all (x) and is such that $H(\xi) = f(\xi)$ on the unit sphere, and further has the property that

$$\int_{|x|=R} |H(x)|^2 dS = O\left(e^{2kCR}\right) ,$$

where C is a non-negative constant. Then there exists a function $U(x)$ which satisfies the radiation condition and the differential equation

$$\Delta U + k^2 U = 0$$

for $|x| > C$; this function is such that

$$U(R\xi) = \frac{e^{ikR}}{R} f(\xi) + O\left(\frac{1}{R^2}\right) .$$

It must be noted here that the constant C appearing in the hypothesis of the theorem gives the radius of the sphere outside of which $U(x)$ is defined. This can be seen to imply that the sources generating the given pattern are located within a sphere of radius C. Therefore the smallest C for which the conditions of our theorem are satisfied determines the radius of the smallest sphere with center at the origin containing all the sources generating a given pattern.

It should be noted that the sources related to a given pattern are not uniquely determined at all; this will be discussed in a subsequent paper. We do show in the present paper, however, that the test formulated in our theorem provides a way of determining the minimum sphere enclosing the sources for a given pattern. For if C_o is the smallest of all numbers C for which the assumptions of our theorem hold, we know that it will not be possible to generate the pattern $f(\xi)$ by sources which are entirely inside a sphere with center at the origin whose radius is smaller than C_o.

We have one immediate consequence of the theorem. Let us assume that $f(\xi)$ vanishes on a portion of the sphere which has non-zero measure. Then from potential theory we know that the harmonic function $H(x)$ vanishes on the whole sphere, so that $f(\xi)$ must be zero everywhere. Therefore we cannot have a radiation pattern which vanishes over any finite angle.

However, if we demand merely that the radiation into that angle be arbitrarily small, we may be able to find such a radiation and we are even able to concentrate the sources at the origin. This could be done by approximating the pattern $f(\xi)$ by spherical harmonics. As is well known, any continuously differentiable function $f(\xi)$ may be represented as a unifromly convergent series of spherical harmonics $S_n(\xi)$ of the order n:

$$f(\xi) = \sum_{n=o}^{\infty} (-i)^n S_n(\xi) \quad .$$

Putting

$$\zeta_n(r) = \sqrt{\frac{\pi}{2r}}\, H^{(1)}_{(2n+1)/2}(kr) \simeq \frac{(-i)^{n+1}}{\sqrt{k}} \frac{e^{ikr}}{r} \quad ,$$

we then find that

$$\sum_{n=0}^{N} \zeta_n(r) S_n(\xi) = U(x)$$

is regular for $|x| > 0$, satisfies the radiation conditions, and possesses the asymptotic expansion

$$U(r\xi) = \frac{e^{ikr}}{\sqrt{k}\, r} \sum_{n=0}^{N} (-i)^{n+1} S_n(\xi) = \frac{-1}{\sqrt{a}} \frac{e^{ikr}}{r} \sum_{n=0}^{N} (-i)^n S_n(\xi),$$

where the pattern

$$f_N(\xi) = \sum_{n=0}^{N} (-i)^n S_n(\xi)$$

gives an approximation of $f(\xi)$ to an arbitrary degree of exactitude if we make N large enough.

Thus the connection between the radius C and the pattern $f(\xi)$ as expressed in our theorem is a purely mathematical relation which is of interest only if we want to represent a certain given pattern exactly.

In the next section we shall obtain the properties of the radiation field of a given pattern. The results are summarized as Theorem 2. In the last section we shall show that these properties are sufficient to determine the field from a given pattern uniquely. These results are summarized in Theorem 3. Theorems 2 and 3 together are equivalent to Theorem 1.

3. The Radiation Pattern of a Given Field

We introduce the following notations. Let

$$\psi_n(r) = \sqrt{\frac{\pi}{2r}}\, J_{(2n+1)/2}(kr) \;;$$

(6)

$$\zeta_n(r) = \sqrt{\frac{\pi}{2r}}\, H^{(1)}_{(2n+1)/2}(kr) \;,$$

where k is fixed. Then we have for $|x| > |y|$ the addition theorem in the form [1]

(7) $$\frac{e^{ik|x-y|}}{|x-y|} = \sum_{n=0}^{\infty} (2n+1) P_n(\xi\eta) \zeta_n(|x|) \Psi_n(|y|) ,$$

where $(\xi\eta)$ is the scalar product of the unit vectors (ξ) and (η) defined by

(8) $$(x) = |x|\,(\xi)\;;\;(y) = |y|\,(\eta)\,.$$

$P_n(t)$ denotes the Legendre polynomial of degree n in t. The series (7), as well as its termwise derivatives with respect to $|y|$, is uniformly convergent for $\frac{|x|}{|y|} \geq \alpha > 1$.

First we shall obtain a representation of a solution of (2) in a series of spherical harmonics.

Lemma 1: Let $U(x)$ be a solution of the equation

$$\Delta U + k^2 U = 0$$

for $|x| \geq C$, which satisfies

$$\lim_{R \to \infty} R\left(\frac{\partial U}{\partial n} - ikU\right) = 0; \quad U(R\xi) = O\left(\frac{1}{R}\right)$$

uniformly for all directions. Then $U(x)$ can be expanded in a series

$$U(x) = \sum_{n=0}^{\infty} \zeta_n(|x|) S_n(\xi) ,$$

where the $S_n(\xi)$ are spherical harmonics of order n. This series converges uniformly for all $|x|$ with $|x| \geq C' > C$.

From Green's theorem we have for any $R > C$ and $R > |x| > C$

(9) $$U(x) = \frac{1}{4\pi} \int_{|y|=C} \left(\frac{e^{ik|x-y|}}{|x-y|} \frac{\partial U}{\partial n} - U \frac{\partial}{\partial n} \frac{e^{ik|x-y|}}{|x-y|} \right) dS_y + \frac{1}{4\pi} \int_{|y|=R} \left(\frac{e^{ik|x-y|}}{|x-y|} \frac{\partial U}{\partial n} - U \frac{\partial}{\partial n} \frac{e^{ik|x-y|}}{|x-y|} \right) dS_y ,$$

where the differentiations and integrations are with respect to y and the derivatives $\frac{\partial}{\partial n}$ are to be taken in the direction of the exterior normals on the boundaries of the region $C \leq |x| \leq R$. Because of the radiation conditions, the second integral vanishes as $R \to \infty$ and we get

$$(10) \qquad U(x) = \frac{1}{4\pi} \int_{|y|=C} \left(\frac{e^{ik|x-y|}}{|x-y|} \frac{\partial U}{\partial n} - U \frac{\partial}{\partial n} \frac{e^{ik|x-y|}}{|x-y|} \right) dS_y .$$

From (7) we find

$$(11) \qquad U(x) = \frac{1}{4\pi} \sum_{\nu=0}^{\infty} (2\nu+1)\zeta_\nu(|x|) \int_{|y|=C} P_\nu(\xi\eta) \left[\Psi_\nu(C) \frac{\partial U}{\partial n} + \Psi'_\nu(C) U \right] dS_y .$$

Since each of the integrals

$$(12) \qquad (2\nu+1) \int_{|y|=C} P_\nu(\xi\eta) \left[\Psi_\nu(C) \frac{\partial U}{\partial n} + \Psi'_\nu(C) U \right] dS_y = S_\nu(\xi)$$

represents a spherical harmonic of order ν, we have proved our Lemma.

From the power series expansion of $\Psi_\nu(C)$ we have for $\nu \to \infty$

$$(13) \qquad \left| \Psi_\nu(C) - \sqrt{\frac{\pi}{2}} \frac{k^{(2\nu+1)/2} C^\nu}{2^{(2\nu+1)/2}} \frac{1}{\Gamma(2\nu+3)/2} \right| \leq \frac{A\left(\frac{kC}{2}\right)^\nu e^{kC/2}}{\Gamma(2\nu+5)/2} ,$$

where C is fixed and A is a suitable constant. This gives the asymptotic relation

$$(14) \qquad \Psi_\nu(C) \simeq \sqrt{\frac{\pi}{2}} \left(\frac{k}{2}\right)^{(2\nu+1)/2} \frac{C^\nu}{\Gamma(2\nu+3)/2} ,$$

which is valid for fixed positive k and C and for $\nu \to \infty$. For our purposes we only need

$$(15) \qquad \Psi_\nu(C) = O\left(\frac{(kC)^\nu}{\Gamma(\nu+1)} \right) .$$

In a similar manner we can also obtain

$$(16) \qquad \Psi'_\nu(C) = O\left(\frac{(kC)^{\nu-1}}{\Gamma(\nu)} \right) .$$

Equation (12) then gives

$$(16a)\qquad |S_\nu(\xi)| \leq A\,\frac{(kC)^\nu}{\Gamma(\nu)} \quad ,$$

where A is a suitable constant independent of ν and (ξ).

For a fixed n and for $r \to \infty$, the $\zeta_n(r)$ have the asymptotic expansion

$$(17)\qquad \zeta_n(r) = \frac{(-i)^{n+1}}{\sqrt{k}}\,\frac{e^{ikr}}{r} + O\left(\frac{1}{r^2}\right) \quad .$$

If we use

$$(18)\qquad U(r\xi) = \sum_{n=o}^{\infty} \zeta_n(r) S_n(\xi)$$

and insert the expansion (17) termwise it would seem that

$$(19)\qquad U(r\xi) = \frac{e^{ikr}}{r}\,\frac{1}{\sqrt{k}} \sum_{n=o}^{\infty} (-i)^{n+1} S_n(\xi) + O\left(\frac{1}{r^2}\right)$$

holds for $r \to \infty$. However, we cannot conclude this result from (17) immediately, since (17) does not hold uniformly with respect to n, and the interchange of limits when passing from (18) to (19) for $r \to \infty$ has to be justified.

Consider the integral representation

$$(20)\qquad \zeta_n(r) = -(-i)^n \sqrt{k} \int_{1+0\,i}^{1+\infty i} e^{ikrt} P_n(t)\,dt \quad .$$

The integral satisfies the same second-order differential equation as $\zeta_n(r)$. Moreover, as it is readily seen after integration by parts, the integrand is singular for $r = 0$ and for $r \to \infty$:

$$(21)\qquad \int_{1+0\,i}^{1+\infty i} e^{ikrt} P_n(t)\,dt = \frac{e^{ikr}}{r} \cdot \frac{i}{k} + O\left(\frac{1}{r^2}\right) \quad .$$

Thus we get (20) by comparing (17) and (21).

Now,

$$(22)\qquad P_n(t) = \frac{1}{2\pi}\int_0^{2\pi}\left(t + i\sqrt{1-t^2}\cos\phi\right)^n d\phi .$$

Since for $0 \leq s$,

$$(23)\qquad |1 + is| + \left|\sqrt{1-(1 + is)^2}\right| \leq (s + 2) ,$$

we have for non-negative s

$$(24)\qquad |P_n(1 + is)| \leq (s + 2)^n .$$

Then, since*

$$(25)\qquad P_n'(t) = (2n - 1)P_{n-1}(t) + (2n - 5)P_{n-3}(t) + \ldots$$

we get from (24)

$$(26)\qquad \begin{aligned} P_n(1 + is) &= 0\left[(is + 2)^n\right] \\ P_n'(1 + is) &= 0\left[n(x + 2)^{n-1}\right] . \end{aligned}$$

These estimates hold uniformly with respect to n and s for all non-negative s. More generally, we have under the same conditions

$$(27)\qquad P_n^{(k)}(1 + is) = 0\left(n^k(s + 2)^{n-k}\right).$$

We form the expression

$$(28)\qquad \phi(t) = -\sqrt{k}\sum_{n=0}^{\infty} S_n(\xi)(-i)^n P_n(t) ,$$

and regard it as a function of (ξ) and t. Because of (16a) this series converges for all t and we have from (26)

$$(29)\qquad \begin{aligned} |\phi(1 + is)| &= 0\left(\sum_{n=0}^{\infty}\frac{[kC(s + 2)]^n}{\Gamma(n)}\right) = 0\left(kCs\, e^{kCs}\right) ; \\ |\phi'(1 + is)| &= 0\left(\sum_{n=0}^{\infty}\frac{n[kC(s + 2)]^{n-1}}{\Gamma(n)}\right) = 0\left((kCs)^2\, e^{kCs}\right) , \end{aligned}$$

*This series ends with $P_0(t)$ if n is odd and with $P_1(t)$ if n is even, so that the series contains less than n terms.

where k and C are fixed and where $\phi'(t) = \frac{d\phi}{dt}$. Equations (29) hold uniformly with respect to (ξ) and s. Let

$$(30) \qquad \phi(t) = -\sqrt{k} \sum_{n=o}^{N} (-i)^n S_n(\xi) P_n(t);$$

then we find from Lemma 1 and (20) that

$$(31) \qquad U(r\xi) = \lim_{N \to \infty} \int_{1+0\,i}^{1+\infty i} e^{ikrt} \phi_N(t)\,dt$$

for $r > C$. Since

$$(32) \qquad |\phi_N(1 + is)| = O\left(kCs\, e^{kCs}\right)$$

uniformly for all n we have proved

<u>Lemma 2:</u> For all $r > 0$,

$$U(r\xi) = \int_{1+0\,i}^{1+\infty i} e^{ikrt} \phi(t)\,dt\ .$$

Our asymptotic expansion is now readily obtained after an integration by parts:

$$(33) \qquad \int_{1+0\,i}^{1+\infty i} e^{ikrt}\phi(t)\,dt = \frac{i}{kr}\, e^{ikr}\phi(1) + \frac{i}{kr} \int_{1+0\,i}^{1+\infty i} e^{ikrt}\phi'(t)\,dt\ .$$

The integral of the right-hand side is of the order $\frac{1}{r^2}$ for $r \to \infty$, as can be seen after another integration by parts.

Since $P_n(1) = 1$ we obtain from (33)

$$(34) \qquad \int_{1+0\,i}^{1+\infty i} e^{ikrt}\phi(t)\,dt = \frac{e^{ikr}}{r} \frac{1}{\sqrt{k}} \sum_{n=o}^{\infty} (-i)^{n+1} S_n(\xi) + O\left(\frac{1}{r^2}\right)\ .$$

This together with Lemma 2 yields Eq. (19). We have therefore proved our asymptotic relation (5) with

$$f(\xi) = \frac{1}{\sqrt{k}} \sum_{n=0}^{\infty} (-i)^{n+1} S_n(\xi) . \tag{35}$$

Now we summarize our results in

<u>Theorem 2:</u> If $U(x)$ is a solution of

$$\Delta U + k^2 U = 0$$

for $|x| > C$ which satisfies the radiation conditions

$$\lim_{R \to \infty} R\left(\frac{\partial U}{\partial n} - ikU\right) = 0; \quad U(R\xi) = O\left(\frac{1}{R}\right)$$

uniformly for all directions, then $U(x)$ can be expanded as

$$U(r\xi) = \sum_{n=0}^{\infty} S_n(\xi)\, \zeta_n(r) \quad ,$$

and this expansion is uniformly convergent for all $r \geq C' > C$. The spherical harmonics $S_n(\xi)$ are such that we may associate with $U(r\xi)$ an integral harmonic function

$$H(r\xi) = H = \frac{1}{\sqrt{k}} \sum_{n=0}^{\infty} (-i)^{n+1} r^n S_n(\xi)$$

such that

$$\int_{|x|=R} |H(r\xi)|^2 dS = O\left(e^{2k(C+\varepsilon)R}\right)$$

for all $\varepsilon > 0$. The asymptotic behavior of U for $r \to \infty$ can be derived from H by

$$U(r\varepsilon) = \frac{e^{ikr}}{r} H(\varepsilon) + O\left(\frac{1}{r^2}\right) .$$

The main part of the result, the estimate for the order of magnitude of H, is obtained as follows. The expressions on the right-hand side of (16a) are coefficients of a power series of an entire function, and this function majorizes.

4. The Field Corresponding to a Given Radiation Pattern

We now prove the converse of Theorem 2.

Theorem 3: Let $H(\mathbf{x})$ be an integral harmonic function and let C and k be positive numbers such that

$$\int_{|\mathbf{x}|=R} |H(\mathbf{x})|^2 \, dS = O\left(e^{2k(C+\varepsilon)R}\right)$$

for all $\varepsilon > 0$. Then there is a uniquely determined function $U(\mathbf{x})$ which satisfies the equation

$$\triangle U + k^2 U = 0$$

and the radiation conditions, and which has the asymptotic expansion

$$U(r\xi) = \frac{e^{ikr}}{r} H(\xi) + O\left(\frac{1}{r^2}\right)$$

for $r \to \infty$.

The harmonic function $H(x)$ can be written as

$$H(r\xi) = \sum_{n=0}^{\infty} r^n S_n(\xi) , \tag{36}$$

where the spherical harmonics $S_n(\xi)$ are uniquely determined.

Put

$$c_n^2 = \int_{|\xi|=1} |S_n(\xi)|^2 \, d\omega . \tag{37}$$

Then

$$\int_{|x|=R} |H(x)|^2 \, dS = R^2 \sum_{n=0}^{\infty} R^{2n} c_n^2 = O\left(e^{2k(C+\varepsilon)R}\right) . \tag{38}$$

Let $S_{n,j}(\xi)$ be a system of $(2n+1)$ orthonormal spherical harmonics of order n. Then any spherical harmonic $S_n(\xi)$ can be expressed in the form

$$S_n(\xi) = \sum_{j=1}^{2n+1} a_j S_{n,j}(\xi) . \tag{39}$$

Moreover[2]

$$\frac{2n+1}{4\pi} P_n(\xi\eta) = \sum_{j=1}^{2n+1} S_{n,j}(\xi) \overline{S_{n,j}(\eta)} , \tag{40}$$

or for $(\xi) = (\eta)$,

$$\frac{2n+1}{4\pi} = \sum_{j=1}^{2n+1} |S_{n,j}(\xi)|^2 . \tag{41}$$

Schwarz's inequality then gives

$$|S_n(\xi)|^2 \leq \frac{2n+1}{4\pi} \sum_{j=1}^{2n+1} |a_j|^2 ; \tag{42}$$

and since the $S_{n,j}(\xi)$ are orthonormal,

$$\sum_{j=1}^{2n+1} |a_j|^2 = \int_{|\xi|=1} |S_n(\xi)|^2 d\omega = C_n^2 . \tag{43}$$

Thus we get

$$|S_n(\xi)| \leq \sqrt{\frac{2n+1}{4\pi}} C_n . \tag{44}$$

Therefore

$$\sum_{n=0}^{\infty} r^n S_n(\xi) = O\left(\sum_{n=0}^{\infty} \sqrt{n} r^n C_n\right) . \tag{45}$$

For any positive $a > 1$ we have

$$\left[\sum_{n=0}^{\infty} \sqrt{n} r^n C_n\right]^2 \leq \sum_{n=0}^{\infty} \frac{n}{a^{2n}} \sum_{n=0}^{\infty} (ar)^{2n} C_n^2 = O\left(e^{2k(C + \varepsilon)ar}\right) , \tag{46}$$

and thus for all $\varepsilon > 0$ we get

$$\sum_{n=0}^{\infty} r^n S_n(\xi) = O\left(e^{k(C + \varepsilon)r}\right) . \tag{47}$$

Therefore

$$-ik \sum_{n=0}^{\infty} S_n(\xi) P_n(1 + is) = \phi(1 + is) \tag{48}$$

$$= O\left(\sum_{n=0}^{\infty} (s + 2)^n S_n(\xi)\right) = O\left(e^{k(C + \varepsilon)s}\right) ,$$

and the function

$$(49) \qquad U(r\xi) = \int_{1+0\cdot i}^{1+\infty i} e^{ikrt}\, \phi(t)\,dt$$

exists for all $r > C$. From

$$(50) \qquad \phi_N(t) = -ik \sum_{n=o}^{N} S_n(\xi) P_n(t)$$

we derive the expression

$$(51) \qquad U_N(r\xi) = \int_{1+0\cdot i}^{1+\infty i} e^{\,ikrt}\, \phi_N(t)\,dt.$$

Each of these functions satisfies

$$(52) \qquad \triangle U_N + k^2 U_N = 0 \;,$$

and the sequence U_n is uniformly convergent in every region $C < a \leq |x| \leq b < \infty$ so that

$$(53) \qquad \lim_{N \to \infty} U_N(x) = U(x) = \int_{1+0\cdot i}^{1+\infty i} e^{ikr}\, \phi(t)\,dt \;.$$

Therefore for $|x| > C$

$$(54) \qquad \triangle U + k^2 U = 0.$$

Because of (48) we have for all $r > C$

$$(55) \qquad \frac{\partial}{\partial r} U(r\xi) = ik \int_{1+0\cdot i}^{1+\infty i} e^{ikrt}\, t\, \phi(t)\; dt \;.$$

By partial integration we thus find from (49) and (55) the asymptotic relations

$$(56) \qquad U(r\xi) = \frac{i}{k}\, \frac{e^{ikr}}{r}\, \phi(1) + O\!\left(\frac{1}{r^2}\right)$$

and

$$\frac{\partial}{\partial r} U(r\xi) = - \frac{e^{ikr}}{r} \dot{\phi}(1) + O\left(\frac{1}{r^2}\right) \qquad (57)$$

This proves that the function $U(r\xi)$ satisfies the radiation condition and also that it has the asymptotic expansion

$$U(r,\xi) = \frac{e^{ikr}}{r} H(\xi) + O\left(\frac{1}{r^2}\right) , \qquad (58)$$

since we have from (48)

$$\dot{\phi}(1) = -ik \sum_{n=o}^{\infty} S_n(\xi) = H(\xi) . \qquad (59)$$

To complete the proof of Theorem 3 it only remains to be shown that $U(x)$ is uniquely determined by the asymptotic expansion (58). Suppose there were two functions satisfying the conditions of Theorem 3. Then their difference U_o would satisfy

$$\triangle U_o + k^2 U_o = 0 \qquad (60)$$

and from (58) it follows that

$$U_o(r\xi) = O\left(\frac{1}{r^2}\right) \qquad (61)$$

uniformly for all directions. From Rellich's Lemma [3] it can be shown that U_o is then identically zero, and thus the uniqueness of $U(x)$ is demonstrated.

[1] Magnus - Obe[illegible] ie speziellen Funktionen der mathemat[illegible] 48. All the formulas are taken from t[illegible]

[2] Ibid, p. 90.

[3] Rellich, F.: [illegible] r Lösungen von $\triangle u + \lambda u = 0$ in unendlic[illegible] sche Mathematiker Vereinigung, 53, 1943, p[illegible]

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